

Most-Perfect Magic Squares

Readers of this column are probably familiar with magic squares. Take, say, the consecutive whole numbers from 1 to 16 and arrange them in a four-by-four array so that every row, every column and the two diagonals all add up to the same total. If you succeed, you've made a magic square of order 4, and the common total is called its magic constant. If you do the same with the numbers 1 to 25 in a five-by-five array, you've created a magic square of order 5, and so on.

Magic squares are a favorite topic of recreational mathematics, and it always seems possible to put a fresh spin on the concept. What is much harder, though, is to make a new contribution to the basic mathematics of the subject. Just such a contribution was published in 1998 by Kathleen Ollerenshaw and David S. Brée in a wonderful book with the slightly

forbidding title *Most-Perfect Pandiagonal Magic Squares: Their Construction and Enumeration*. (The publisher is the Institute of Mathematics and Its Applications, Southend-on-Sea, England.)

The book presents the first significant partial solution of one of the biggest unsolved problems in the subject: to count how many magic squares there are of any given order. The main result is a formula for the number of so-called most-perfect squares, a special subset of magic squares with particularly remarkable properties. In case this sounds like an easy problem, it is worth pointing out that the number of such squares of order 12 is more than 22 billion. The number for order 36 is roughly 2.7×10^{44} . Obviously, you can't count these squares by writing them out individually.

Ollerenshaw and Brée tackled the problem using an area of mathematics

known as combinatorics, which is the art of counting things "by the back door"—that is, without listing them. A noteworthy feature of the research is that neither author is a typical mathematician. Ollerenshaw, who is 87, spent much of her professional life as a high-level administrator for several English universities. Brée has held university positions in business studies, psychology and, most recently, artificial intelligence.

For mathematical purposes, it is convenient to build a magic square of order n from the integers $0, 1, 2, \dots, n^2 - 1$, and both the book and this column employ that convention. Traditional magic squares, however, do not include 0; instead they use the integers $1, 2, 3, \dots, n^2$. There is no essential difference between the two conventions—if you add 1 to every entry in a mathematician's magic square, you get a traditional square, and conversely if you subtract 1 from every entry in a traditional magic square, you get a mathematician's square. The only thing that changes is the square's magic constant, which is increased or diminished by n .

There is a single magic square of order 1, namely, the number 0 standing alone. There is no magic square of order 2 (the only order that never occurs), because the conditions force all four entries to be equal. There are eight magic squares of order 3, but they are all rotations or reflections of just one square with a magic constant of 12:

1	8	3
6	4	2
5	0	7

A rotation or a reflection of a magic square remains magic, so all magic squares of order 3 are essentially the same. There are lots of different magic squares of order 4, however, and the number explodes as the order increases. No exact formula is known.

One way to make progress is to impose further conditions on the magic squares. For our purposes, the most natural such condition is that the square should be pandiagonal—all the square's

64	92	81	94	48	77	67	63	50	61	83	78
31	99	14	97	47	114	28	128	45	130	12	113
24	132	41	134	8	117	27	103	10	101	43	118
23	107	6	105	39	122	20	136	37	138	4	121
16	140	33	142	0	125	19	111	2	109	35	126
75	55	58	53	91	70	72	84	89	86	56	69
76	80	93	82	60	65	79	51	62	49	95	66
115	15	98	13	131	30	112	44	129	46	96	29
116	40	133	42	100	25	119	11	102	9	135	26
123	7	106	5	139	22	120	36	137	38	104	21
124	32	141	34	108	17	127	3	110	1	143	18
71	59	54	57	87	74	68	88	85	90	52	73

ILLUSTRATIONS BY SARAH L. DONELSON

MAGIC SQUARE
of order 12 is most-perfect because the numbers in any two-by-two block (black squares) add up to the same total: 286.

“broken diagonals” must also sum to the magic constant. (Broken diagonals wrap around from one edge of the square to the opposite edge.) An example of a pandiagonal magic square with a magic constant of 30 is:

0	11	6	13
14	5	8	3
9	2	15	4
7	12	1	10

Examples of the broken diagonals here are $11 + 8 + 4 + 7$ and $11 + 14 + 4 + 1$, both of which do indeed add up to 30. The order 3 square is not pandiagonal: for example, $8 + 2 + 5 = 15$, not 12. In fact, a magic square cannot be pandiagonal unless its order is doubly even—that is, a multiple of 4.

Most-perfect squares are even more restricted. As well as being magic and pandiagonal, they also have the property that any two-by-two block of adjacent entries sum to the same total, namely, $2n^2 - 2$, where n is the order. (It can also be shown that any magic square with this two-by-two property is necessarily pandiagonal.) The order 4 square shown above is most-perfect—for example, the entries in the two-by-two block consisting of 0, 11, 14 and 5 add up to 30. Note that we include two-by-two blocks that wrap around from one edge of the square to the opposite edge, such as the block consisting of 3, 4, 14 and 9. More ambitiously, the order 12 square shown on the opposite page is also most-perfect.

The key to Ollerenshaw and Brée’s counting method is a connection between most-perfect squares and “reversible squares.” To explain what these are, we need some terminology. A sequence of integers has reverse similarity if, when the sequence is reversed and the corresponding numbers are added, the totals are all the same. For example, the sequence 1, 4, 2, 7, 5, 8 has reverse similarity, because its reversal is 8, 5, 7, 2, 4, 1 and the sums of the corresponding numbers— $1 + 8$, $4 + 5$, $2 + 7$, $7 + 2$, $5 + 4$ and $8 + 1$ —are all equal to 9.

A reversible square of order n is an n -by- n array formed by the integers 0, 1, 2, ..., $n^2 - 1$ with the following properties: every row and column have reverse similarity, and in any rectangular array of integers from the square, the sums of

entries in opposite corners are equal. For instance, the four-by-four array of the integers 0 to 15 in ascending order is reversible, as shown below. In the third row, for example, $8 + 11 = 9 + 10 = 19$. The same pattern holds for all other rows and all columns. Moreover, equations such as $5 + 11 = 7 + 9$ and $1 + 15 = 3 + 13$ verify the second condition:

0	1	2	3
4	5	6	7
8	9	10	11
12	13	14	15

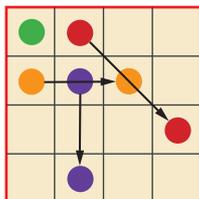
Reversible squares are generally not magic, but Ollerenshaw and Brée prove that every reversible square of doubly even order can be changed to a most-perfect magic square by a specific procedure and that every most-perfect magic square can be produced in this manner. We show the procedure on the order 4 reversible square above. First, reverse the right-hand half of each row:

0	1	3	2
4	5	7	6
8	9	11	10
12	13	15	14

Reverse the bottom half of each column:

0	1	3	2
4	5	7	6
12	13	15	14
8	9	11	10

Now break up the square into two-by-two blocks. Move the four entries in each block as shown below:



That is, the top left entry stays fixed, the top right moves diagonally two squares, the bottom left moves two spaces to the right, and the bottom right moves two spaces down. If any number

falls off the edge of the four-by-four square, wrap the edges around the square to find where it should go. (This particular method works only for order 4 squares. For the general case of order n , there is a similar recipe expressed by a mathematical formula.) The result here is a most-perfect magic square:

0	14	3	13
7	9	4	10
12	2	15	1
11	5	8	6

The transformation process sets up a one-to-one correspondence between most-perfect magic squares and reversible squares of doubly even order. Therefore, you can count the number of most-perfect magic squares by counting the number of reversible squares of the same order. At first sight, this change in the nature of the problem doesn’t seem to get you very far, but it turns out that reversible squares have several nice features that make it possible to count them.

In particular, reversible squares fall into classes. Within each class, all members are related to one another by a variety of transformations, such as rotations, reflections and a few more complicated maneuvers. To construct all members of such a class, it is enough to construct one of them and then routinely apply the transformations. Furthermore, each class contains precisely one “principal” square. Finally, the size of each class is the same. The number of essentially different squares in each class is precisely $2^{n-2}((n/2)!)^2$, where the exclamation point indicates a factorial. (For example, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$.)

It thus remains only to count how many principal reversible squares there are of a given order and to multiply that number by the above formula. The result will be the number of essentially different most-perfect magic squares of that order. It turns out that the number of principal reversible squares can itself be given as a formula, though a rather complex one. The discovery of this formula, and its proof, leads deeper into combinatorics, so I’ll stop here, except to say that for the doubly even orders $n = 4, 8, 12$ and 16, the numbers of different most-perfect magic squares are 48, 368,640, 2.22953×10^{10} and 9.322433×10^{14} . SA